ON THE TERNARY BI-QUADRATIC DIOPHANTINE EQUATION

$$3(X^2+Y^2)-5XY=36Z^4$$

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Abstract:

The ternary biquadratic non-homogeneous equation represent by the Diophantine equation $3(X^2+Y^2)-5XY=36Z^4$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special numbers are exhibited.

KEYWORDS:

Integral solutions, ternary biquadratic non-homogeneous equation, lattice points.

MSC 2010 mathematics subject classification:11D25

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INTRODUCTION:

The biquadratic Diophantine (homogeneous or non-homogeneous) equations offer an unlimited field for research due to their variety [1-3].In particular, one may refer [4-15] for ternary non-homogeneous bi-quadratic equations. This communication concerns with yet another interesting ternary non-homogeneous biquadratic equation given by $3(X^2+Y^2)-5XY=36Z^4$ for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS:

- $T_{m,n}$: Polygonal number of rank n with size m.
- $CP_{30,n}$: Centered Triacontagonal Pyramidal number of rank 30.
- $CP_{28,n}$: Centered Icosioctagonal Pyramidal number of rank 28.
- $CP_{24,n}$: Centered Icositeragonal Pyramidal number of rank 24.
- $CP_{20,n}$:Centered Icosogonal Pyramidal number of rank 20.
- $CP_{18,n}$:Centered Octadecagonal Pyramidal number of rank 18.
- $CP_{10,n}$:Centered Decagonal Pyramidal number of rank 10.
- $CP_{6,n}$:Centered Hexagonal Pyramidal number of rank 6.
- $F_{4,n,3}$: Four dimensional figurate number of rank 3.
- $F_{4,n,4}$: Four dimensional figurate number of rank 4.

METHOD OF ANALYSIS:

The ternary biquadratic equation to be solved for its non-zero integer solution is,

$$3(X^2+Y^2)-5XY=36Z^4 \qquad (1)$$

Let $x=u+v, y=u-v, u\neq v\neq 0$ (2)

Substituting (4) in (1) and simplifying, we get

$$u^2+11v^2=36z^4 \qquad (3)$$

We present below different methods of solving (3) and thus obtained different patterns of solution to (1).

PATTERN: 1

Assume, $z(a,b) = a^2 + 11b^2$ (4)

Write 36 as

$36 = (5 + i\sqrt{11})(5 - i\sqrt{11}$ (5)

Substituting (4) and (5) in (3) and employing the method of factorization, define

$$(u + i\sqrt{11}v) = (5 + i\sqrt{11})(a + i\sqrt{11}b)^4$$

Equating real and imaginary parts, weget

$$u = 5a^4 + 605b^4 - 330a^2b^2 - 44a^3b + 484ab^3$$

$$v = a^4 + 121b^4 - 66a^2b^2 + 20a^3b - 220ab^3 \quad (6)$$

Using (6) in (2) wehave,

$$\left.\begin{aligned} x &= 6a^4 + 726b^4 - 396a^2b^2 - 24a^3b + 264ab^3 \\ y &= 4a^4 + 484b^4 - 264a^2b^2 - 64a^3b + 704ab^3 \end{aligned}\right\} \quad (7)$$

Thus (7) and (4) represent the non-zero distinct integral solutions of (1).

PROPERTIES:

- $144F_{4,a,3} - x(a,1) - 6CP_{30,a} - 30CP_{6,a} - T_{926,a} \equiv 1(\bmod 2)$
- $96F_{4,a,3} - y(a,1) - 6CP_{28,a} - 60CP_{6,a} - T_{618,a} \equiv 1(\bmod 4)$
- $17424F_{4,b,3} - x(b,1) - 6CP_{30,b} - 4062CP_{6,a} - T_{16766,b} \equiv 2(\bmod 3)$
- $y(a,1) + 6CP_{30,a} + 24F_{4,a,4} - x(a,1) - T_{298,a} \equiv 1(\bmod 2)$

PATTERN: 2

Equation (3) can be written as,

$$u^2 - 25z^4 = 11(z^4 - v^2) \tag{8}$$

Equation (8) is rewritten in the form of ratio as,

$$\frac{u+5z^2}{z^2+v} = \frac{11(z^2-v)}{u-5z^2} = \frac{A}{B}, B \neq 0$$

Which is equivalent to the following two equations,

$$\left.\begin{aligned} Bu - Av + z^2(5B - A) = 0 \\ Au + 11Bv + z^2(-5A - 11B) = 0 \end{aligned}\right\} \tag{9}$$

Employing the method of cross multiplication to solve the system of (9), weget

$$\left.\begin{aligned} u = 5A^2 - 55B^2 + 22AB \\ v = -A^2 + 11B^2 + 10AB \end{aligned}\right\} \tag{10}$$

$$z^2 = A^2 + 11B^2 \tag{11}$$

Taking $A = p^2 - 11q^2$, $B = 2pq$ in (10) and (9) weget,

$$\left.\begin{aligned} u = 5p^4 + 605q^4 - 330p^2q^2 + 44p^3q - 484pq^3 \\ v = p^4 + 121q^4 - 66p^2q^2 - 20p^3q + 220pq^3 \end{aligned}\right\} \tag{12}$$

$$z = p^2 + 11q^2 \tag{13}$$

Using the values of u and v from (12) in (2) we've,

$$\left.\begin{aligned} x = 6p^4 + 726q^4 - 396p^2q^2 + 24p^3q - 264pq^3 \\ y = 4p^4 + 484q^4 - 264p^2q^2 + 64p^3q - 704pq^3 \end{aligned}\right\} \tag{14}$$

Thus (13) and (14) represent the non-zero integral solutions of (1).

PROPERTIES:

- $96F_{4,p,3} + 6CP_{30,p} - y(p,1) + 6CP_{30,p} - T_{618,p} \equiv 3 \pmod 4$

- $144F_{4,p,3} - x(p,2) + 12CP_{6,p} - T_{3302,p} \equiv 5(\bmod 6)$
- $6CP_{3,p} + 6CP_{20,p} - 24F_{4,p,4} - y(p,1) + x(p,1) + T_{298,p} \equiv 1(\bmod 2)$
- $11616F_{4,q,3} - 6CP_{30,q} - y(2,q) - 4282CP_{6,q} - T_{12762,q} \equiv 4(\bmod 8)$

PATTERN: 3

Equation (8) is also written in the form of ratio as,

$$\frac{u+5z^2}{11(z^2+v)} = \frac{z^2-v}{u-5z^2} = \frac{A}{B}, B \neq 0$$

Which is equivalent to the following two equations,

$$\left.\begin{aligned} Bu - 11Av + z^2(5B-11A) = 0 \\ Au + Bv - z^2(5A+B) = 0 \end{aligned}\right\} \qquad (15)$$

Employing the method of cross multiplication to solve the system of (15), weget

$$\left.\begin{aligned} u &= 55A^2 - 5B^2 + 22AB \\ v &= 11A^2 - B^2 - 10AB \end{aligned}\right\} \qquad (16)$$

$$z^2 = 11A^2 + B^2 \qquad (17)$$

Taking $A = 2pq$, $B = 11p^2 - q^2$ in (17) and (16) weget,

$$\left.\begin{aligned} u &= 605p^4 + 5q^4 - 330p^2q^2 - 484p^3q + 44pq^3 \\ v &= 121p^4 + q^4 - 66p^2q^2 + 220p^3q - 20pq^3 \end{aligned}\right\} \qquad (18)$$

$$z = 11p^2 + q^2 \qquad (19)$$

Using the values of u and v from (18) in (2) we've,

$$\left.\begin{aligned} x &= 726p^4 + 6q^4 - 396p^2q^2 - 264p^3q + 24pq^3 \\ y &= 484p^4 + 4q^4 - 264p^2q^2 - 704p^3q + 64pq^3 \end{aligned}\right\} \qquad (20)$$

Thus (19) and (20) represent the non-zero integral solutions of (1).

PROPERTIES:

- $6CP_{24,q} + 144F_{4,q,3} - x(1,q) - T_{794,q} \equiv 2(\bmod 3)$
- $96F_{4,q,3} - y(1,q) + 6CP_{3o,q} + 6CP_{24,q} + 3CP_{10,q} - T_{530,q} \equiv 1(\bmod 2)$
- $240F_{4,q,3} - x(1,q) - y(1,q) + 6CP_{30,q} + 58CP_{6,q} - T_{1322,q} \equiv 3(\bmod 5)$
- $144F_{4,q,3} - x(2,q) + 6CP_{30,q} + 6CP_{18,q} - T_{3170,q} \equiv 5(\bmod 6)$

CONCLUSION:

To conclude, one may search for other patterns of solutions and their corresponding properties.

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